AD-A061 061   NAVAL POSTGRADUATE SCHOOL MONTEREY CALIF   F/G 14/4
ANALYTICAL HAZARD REPRESENTATION FOR USE IN RELIABILITY, MORTAL--ETC(U)
AUG 78   D P GAVER, M ACAR
UNCLASSIFIED   NPS55-78-017   NL

| OF |
AD
A061 061

END
DATE
FILMED
1-79
DDC

2 LEVEL II

AD A061061

DDC FILE COPY

NPS55-78-017

NAVAL POSTGRADUATE SCHOOL

Monterey, California

9 Technical rept.

6 ANALYTICAL HAZARD REPRESENTATION FOR USE IN RELIABILITY, MORTALITY, AND SIMULATION STUDIES. Part I.

by

10 D. P./Gaver

~~and~~

M./Acar

14 NPS55-78-017

16 RR01405

17 RR0140501

11 August 1978

12 49p.

Approved for public release; distribution unlimited.

Prepared for:
Chief of Naval Research, Arlington, Va. 22217

251 450

78 10 30 045

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

Rear Admiral T. F. Dedman
Superintendent

J. R. Borsting
Provost

Research supported in part by the National Science Foundation under Grant MCS-77-07587 and the Office of Naval Research.

This report was prepared by:

D. P. Gaver

D. P. Gaver, Professor
Department of Operations Research

M. Acar
Department of Operations Research

Reviewed by:

Michael Sovereign

Michael G. Sovereign, Chairman
Department of Operations Research

Released by:

William M. Tolles

William M. Tolles
Dean of Research

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NPS55-78-017	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) Analytical Hazard Representation for Use in Reliability, Mortality, and Simulation Studies Part 1.		5. TYPE OF REPORT & PERIOD COVERED Technical
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) D. P. Gaver and M. Acar		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, Ca. 93940		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61153N, RR014-05-01, NR-042-363; N0001478WR80009
11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217		12. REPORT DATE August 1978
		13. NUMBER OF PAGES 48
14. MONITORING AGENCY NAME & ADDRESS(if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Reliability, availability, hazard, data analysis, simulation.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

A simple parametric model is proposed to represent data of non-standard distributional form. An example is the "bath tub" hazard of reliability. The application of the approach to simulation and to data analysis is discussed and will be further explored in later reports.

TABLE OF CONTENTS

ANALYTICAL HAZARD REPRESENTATIONS FOR USE IN RELIABILITY, MORTALITY, AND SIMULATION STUDIES

Part I

D. P. Gaver *

M. Acar

I. INTRODUCTION

The failure rate function, or hazard function (hazard for short) may be described as the conditional probability of an equipment's failing at operating age t, having survived to that age. The reliabilities of a variety of electronic and mechanical items are conveniently and naturally described in terms of the appropriate hazard function, and so is the longevity of human beings. The term force of mortality replaces hazard in the latter context.

This paper is devoted to a study of several simple analytical representations for hazard functions. These representations are in turn based upon representations of random variables having certain required properties, in terms of others having familiar distributions--in particular the exponential. Similar ideas are due to Tukey (1976) and recently have been examined by Parzen (1978). The hazard representations proposed are quite

* Research sponsored by the Office of Naval Research.

expeditiously used in simulation studies, e.g. of system reliability or availability in terms of component lifetimes. They may also be used in data analysis studies, in order to parsimoniously describe data sets in terms of perturbations of convenient and familiar standard distributions. Their use in data analysis and simulation is also described in Gaver, Lavenberg, and Price (1976), and in Gaver and Chu (1977).

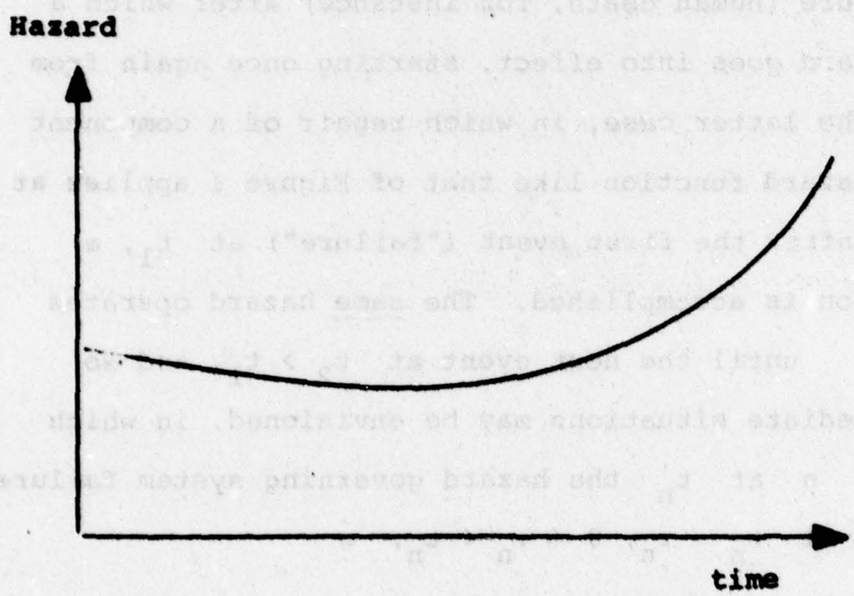
II. SYSTEM FAILURE PATTERNS

It is plausible to think that the time series of failures in a system may involve these stages.

1. Early failures. There may be a relatively large number of failure soon after a system is introduced because of design defects, production errors, or errors stemming from maintenance personnel inexperience. This situation is characterized by a hazard function that is initially large, but that decreases with time. "Infant mortality" is in evidence.
2. Random Failures. Following the early failure period there may be a period during which failures occur at an essentially constant rate for a rather prolonged time. During this period the hazard function is nearly constant, so the times between failures are close to being exponentially distributed. The effect of age or wearout is not yet apparent.

3. <u>Wearout Failures</u>. Eventually following the period during which a constant hazard is evident there is likely to be a period of ever-increasing failure rate caused by wearout of system components.

A graphical representation of a hazard function that exhibits the behavior described is given below. Note that it has the legendary "bath tub" shape.



Some comments on the above follow:

A. The term "failure" may refer to an event that is analogous to human death, after which the entire system is replaced. On the other hand repair or component replacement may occur after failure; the system is only repaired, not entirely replaced. In the former case, a hazard function of the kind depicted in Figure 1 applies to each system event ("death"); when the system is installed (or is born) that hazard operates starting from scratch at $t = 0$ until system failure (human death, for instance) after which a similar hazard goes into effect, starting once again from zero. In the latter case, in which repair of a component occurs, a hazard function like that of Figure 1 applies at $t = 0$, but after the first event ("failure") at t_1, a repair action is accomplished. The <u>same</u> hazard operates for $t \geq t_1$ until the next event at $t_2 > t_1$, and so on. Intermediate situations may be envisioned, in which after event n at t_n the hazard governing system failure $n+1$ starts at $t_n - \tau_n$, $0 < \tau_n < t_n$.

B. Although there is reason to assume that hazards somewhat like that of Figure 1 occur in general for systems, the possibility exists that the system hazard is "bumpy" because wearout failures of components or subsystems may well occur at intermediate times.

C. If the theory is applied to systems with little or no wearout propensity, as should be the case when dealing with computer software modules, then the hazard function may well exhibit the initial falloff of Figure 1 but not the rise at later times. In fact, a constant decline as bugs are found and removed could be (optimistically) anticipated for software. The right-hand side of the bath tub vanishes, and the picture is that of a ski slope.

III. MODELS FOR THE HAZARD FUNCTION

In this section mathematical models are presented for the failure rate or hazard function. Recall that the hazard may be defined as follows.

Definition. Suppose that the time to failure, X, is a random variable with distribution function $F(x)$, where $F(0) = 0$; the latter possesses the density function $f(x)$, $f(x) = dF/dx$, such that for any positive x,

$$F(x) = \int_0^x f(y)dy . \tag{3.1}$$

Then the hazard function, or failure rate at age x, is given by

$$h(x) = \frac{f(x)}{1 - F(x)} . \tag{3.2}$$

The interpretation of $h(x)dx$ is that it is the conditional probability of failure in the interval $(x, x + dx)$, given that there has been no failure up to age x.

Express the hazard as

$$h(x) = \frac{dF/dx}{1 - F(x)}$$

$$h(x)dx = \frac{dF}{1 - F(x)} = - d\{\log[1 - F(x)]\},$$

it then follows after integration that

$$F(x) = 1 - \exp[- \int_0^x h(y)dy] \; . \qquad (3.3)$$

Thus if the hazard is specified, so is the distribution function, and conversely.

Note that if

$$h(x) = \lambda > 0 \; , \qquad 0 < x$$

then

$$F(x) = 1 - e^{-\lambda x}, \qquad 0 \leq x \; , \qquad (3.4)$$

so a constant hazard function implies the exponential distribution of the random variable X, and conversely.

Obviously a constant hazard representation does not describe the bath tub hazard shape of Figure 1, nor does it represent a situation in which hazards decline, possibly because design defects or "bugs" are occasionally removed. Here are two hazard representations likely to be useful for such purposes.

(1) A Bath tub Model

Define the random variable Z in terms of X, X being exponentially distributed with mean λ^{-1}, as follows:

$$Z = G(X) = XL(X)\ R(X)$$

or (3.5)

$$= X\phi(X), \quad \phi(X) = L(X)R(X)$$

where

a) $L(x)$ is convex in x, $L(0) < 1$, $L(\infty) = 1$,

b) $R(x)$ is concave in x, $R(0) = 1$, $R(0) > R(\infty)$.

Then the hazard of Z may be made to exhibit a bath tub shape, as in Figure 1, by proper choice of the function L and R.

Example. Suppose

$$L(x) = \frac{\alpha x}{1 + \alpha x} \qquad \alpha > 0,\ 0 \leq x$$

(3.6)

$$R(x) = \frac{1}{1 + \beta x} \qquad \beta > 0\ ,$$

Clearly

$$z = xL(x)\ R(x) = x \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x} = \frac{\alpha x}{1 + \alpha x} \quad \frac{x}{1 + \beta x}$$

is a monotonically increasing function of x. Furthermore, choose α large (e.g. $\alpha = 10$) and β small (e.g. $\beta = 10^{-3}$). Then it is intuitively clear that (i) small x-values transform into even smaller z-values, e.g. $x = 1$ corresponds to $z = 0.91$ and $x = 2$ corresponds to $z = 1.90$, but (ii) this effect dwindles as x increases, so $x = 10$ corresponds to $z = 9.8$ and $x = 50$ to $z = 47.5$ and the z-values closely resemble the x's percentage-wise, but (iii) as x increases still further the z's do not follow suit: $x = 10^3$ corresponds to $z = 500$. This suggests that if x is a value assumed by X, that Z shares the properties of X in mid-range, i.e. for intermediate x-values, but differs from X by having a disproportionate probability of assuming small values (near zero), or large values (near, but less than, $1/\beta$). Thus the hazard of Z will appear to be a "bath tubbed" version of X, particularly if X is exponential.

We focus attention on the representation (3.6) in what follows, mainly for analytical and computational convenience. Of course there are many other possibilities, such as

$$L(x) = 1 - e^{-\alpha x}$$

$$R(x) = e^{\beta x} \; ; \qquad (3.7)$$

these latter may be adjusted to provide sharper-edged bath tubs than can (3.6), but iteration of (3.6) may be induced to accomplish the same purpose.

(2) A Decreasing Failure Rate Model

Define the random variable W in terms of X, X again being exponential with parameter λ^{-1}:

$$W = XT(X) \qquad (3.8)$$

where $T(x)$ is an increasing function of x, $L(0) = 1$. Then the hazard may be made to exhibit a decreasing behavior.

Example. Suppose

$$T(x) = 1 + cx \; , \qquad c > 0, \; 0 \leq x. \qquad (3.9)$$

Then

$$z = x(1 + cx) \qquad (3.10)$$

is monotonic, and small x-values lead to comparable z-values (especially when c is small), but larger x-values are "amplified" by $1 + cx$ to yield increasingly large z-values.

Attention will be focused upon (3.9), although other possibilities exist that accomplish the same purpose, namely that of lengthening the right tail of the distribution of X (simulating outliers, for instance) while leaving the body of the distribution virtually unchanged.

IV. MATHEMATICAL PROPERTIES OF THE "BATH TUB" HAZARD MODEL

Various analytical properties of the previously described models will now be recorded. These provide useful insights into the behavior of the random variables Z and the underlying (generating) variables X.

A. Monotonicity; Quantiles

It is convenient to focus on montonic increasing transformations, i.e. if

$$z = G(z) = z\phi(z) \tag{4.1}$$

then in order that the above function be monotonically increasing, $dz/dx > 0$. Observe that logarithmic differentiation of (4.1) provides

$$\frac{dz}{z} = \frac{dx}{x} + \frac{\phi'(x)}{\phi(x)}\, dx \tag{4.2}$$

and thus $dz/dx > 0$ if and only if

$$\frac{1}{x} + \frac{\phi'(x)}{\phi(x)} > 0 \tag{4.3}$$

Alternatively, the condition is, in terms of $L(x)$ and $R(x)$,

$$\frac{1}{x} + \frac{L'(x)}{L(x)} + \frac{R'(x)}{R(x)} > 0 \tag{4.4}$$

It is easily seen that the important example (3.6),

$$\phi(x) = \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x},$$

yields a monotonic relationship between z and x. The fact that this transformation can be easily and explicitly <u>inverted</u> (solved for x in terms of z) will be exploited subsequently.

Of course if $z(x)$ is monotonically increasing then so is $x(z)$, the inverse function. The events $(Z \leq z)$ and $(X \leq x(z))$ are equivalent, and so

$$P\{Z \leq z\} = P\{X \leq x(z)\}, \tag{4.5}$$

from which it follows that if $x_p \equiv x(p)$ is the p·100% quantile of X, i.e.

$$P\{X \leq x(p)\} = p, \tag{4.6}$$

then

$$P\{Z \leq z(p)\} = P\{Z \leq z(x(p))\} = p \tag{4.7}$$

and so $z(p)$, the $p \cdot 100\%$ quantile of Z is simply obtained from

$$z(p) = x(p)\ \phi(x(p)) = x(p)\ L(x(p))\ R(x(p)) \qquad (4.8)$$

In other words we very easily translate from (points on) the inverse distribution of X to the inverse distribution of Z. Explicit representation of the distribution of Z is however, not often easily possible.

B. Hazard and Density Function Relationships

In order to investigate the relationship between the hazards of Z and X, begin by writing

$$p = F_X(x(p)) = 1 - \exp[-\int_0^{x(p)} h_x(u)du] \qquad (4.9)$$

or

$$\int_0^{x(p)} h_x(u)du = -\ell n(1-p)$$

Now differentiate with respect to p to find

$$h_x(x(p))\ \frac{dx(p)}{dp} = \frac{1}{1-p} \qquad (4.10)$$

or

$$h_x(x(p)) = \frac{dp}{dx(p)} \cdot \frac{1}{1-p} = f_x(x(p)) \cdot \frac{1}{1-p}\ ; \qquad (4.11)$$

here h_x and f_x are the hazard and density functions of the r.v. X. The relationship (4.11) holds for any distribution, of course.

Differentiation of (3.5) reveals the connection between h_z and h_x. From (4.2)

$$\frac{dz(p)}{dp} = z(p)\left[\frac{1}{x(p)} + \frac{\phi'(x(p))}{\phi(x(p))}\right]\frac{dx(p)}{dp} \tag{4.12}$$

$$= [\phi(x(p)) + x(p)\ \phi'(x(p))]\ \frac{dx(p)}{dp}$$

From (4.11), applied now to the z-hazard, there results

$$\frac{1}{h_z(z(p))} = \frac{1}{h_x(x(p))}\ [\phi(x(p)) + x(p)\ \phi'(x(p))]\ , \tag{4.13}$$

so

$$h_z(z(p)) = h_x(x(p))\ \frac{1}{\phi(x(p)) + x(p)\ \phi'(x(p))} \tag{4.14}$$

Multiplication of both sides by $1-p$ then shows, in view of (4.11), that the density functions are similarly related:

$$f_z(z(p)) = f_x(x(p))\ \frac{1}{\phi(x(p)) + x(p)\ \phi'(x(p))}$$

<u>Example</u>.

X is exponential(λ). Then

$$h_z(x_p) = \frac{\lambda}{\phi(x(p)) + x(p)\ \phi'(x(p))} \tag{4.15}$$

Now use the specific $\phi(x)$ of (3.6):

$$\phi(x) = \frac{\alpha x}{1 + \alpha x} \cdot \frac{1}{1 + \beta x}$$

or, in terms of logarithms,

$$\ell n\ \phi(x) = \ell n\ \alpha x - \ell n(1 + \alpha x) - \ell n(1 + \beta x) ,$$

so

$$\frac{\phi'(x)}{\phi(x)} = \frac{1}{x} - \frac{\alpha}{1 + \alpha x} - \frac{\beta}{1 + \beta x} = \frac{1 - \alpha\beta x}{x(1 + \alpha x)(1 + \beta x)} , \tag{4.16}$$

and

$$\phi(x) + x\phi'(x) = \phi(x) \left[\frac{2 + (\alpha + \beta)x}{(1 + \alpha x)(1 + \beta x)} \right] ; \tag{4.17}$$

finally

$$h_z(z(p)) = \frac{\lambda(1 + \alpha x(p))^2 \ (1 + \beta x(p))^2}{\alpha x(p) \ [2 + (\alpha + \beta) \ x(p)]} \tag{4.18}$$

Although this expression is not quite explicit, qualitative properties of h_z can be deduced from it.

(a) If $p \to 0$, $x(p) = -\frac{1}{\lambda} \ \ell n(1-p) \downarrow 0$, and hence

$$h_z(z(p)) \sim \frac{\lambda}{2\alpha \ x(p)} , \tag{4.19}$$

or

$$\lim_{p \to 0} \ x(p) \ h_z(z(p)) = \frac{\lambda}{2\alpha} \tag{4.20}$$

Since for $p \to 0$,

$$z(p) \sim \alpha x^2(p)$$

and hence

$$x(p) \sim [z(p)/\alpha]^{1/2} \qquad (4.21)$$

there results

$$h_z(z(p)) \sim \frac{\lambda}{2[z(p)]^{1/2}\sqrt{\alpha}}$$

or

$$\lim_{p \to 0} \sqrt{z(p)}\, h_z(z(p)) = \frac{\lambda}{2\sqrt{\alpha}} \qquad (4.22)$$

This shows that $h_z(z) \uparrow \infty$ as $z \to 0$, creating the left-hand end of the bath tub of Figure 1.

(b) If $p \to 1$, $x(p) \uparrow \infty$, and $z(p) \uparrow 1/\beta$ so

$$h_z(z(p)) \sim \lambda \frac{\alpha\beta^2 x^2(p)}{(\alpha + \beta)}$$

or

$$\lim_{p \to \infty} \frac{1}{[x(p)]^2} h_z(z(p)) = \lambda \frac{\alpha\beta^2}{\alpha+\beta} \qquad (4.23)$$

For $p \to 1$

$$1 - \beta z(p) \sim \frac{(\alpha + \beta)}{\alpha\beta} \frac{1}{x(p)} \qquad (4.24)$$

so

$$x(p) \sim \frac{\alpha + \beta}{\alpha\beta} \frac{1}{1 - \beta z(p)} \qquad (4.24)$$

and thus

$$h_z(z(p)) \sim \lambda\left(\frac{\alpha + \beta}{\alpha}\right) \frac{1}{(1 - \beta z)^2} \tag{4.26}$$

Once again it appears that the hazard rises rapidly, this time as $x(p) \uparrow \infty$ and $z(p) \uparrow \beta^{-1}$; the other end of the bath tub is thus fashioned.

(c) If $p = 1 - e^{-1}$, then $x(p) = \lambda^{-1}$. Then

$$h_z(z(1-e^{-1})) = \frac{\lambda[1 + \alpha/\lambda]^2 \; [1 + \beta/\lambda]^2}{\frac{\alpha}{\lambda} \; [2 + (\alpha + \beta)/\lambda]} \tag{4.27}$$

The bath tub effect is presumably achieved by choosing α large and β small. *Let* $\alpha \rightarrow 0$ *and* $\beta \rightarrow 0$ *independently in* (4.27); it is clear that the limiting value of the hazard is λ. This indicates that the hazard is (approximately) λ for middling values of z.

C. An Explicit Formula for a Hazard

The expression (3.6) leads to the relationship

$$z(p) = \frac{\alpha x^2(p)}{(1 + \alpha x(p))(1 + \beta x(p))}, \tag{4.28}$$

and the latter may be explicitly inverted by solving a quadratic equation. The result is

$$x(p) = \frac{(\alpha + \beta)\ z(p) + \sqrt{(\alpha + \beta)^2\ z^2(p) + 4z(p)\ \alpha(1 - \beta z(p))}}{2\alpha(1 - \beta z(p))} \tag{4.29}$$

Now a direct differentiation of this expression and invocation of (4.11) produces the expression

$$h_z(z) = \frac{\lambda}{2\alpha(1-\beta z)} \left[\frac{\beta\{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}\}}{1 - \beta z} + (\alpha + \beta) \right.$$

$$\left. + \frac{\{(\alpha-\beta)^2 z + 2\alpha\}\ \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{(\alpha + \beta)^2 z^2 + 4\alpha z(1 - \beta z)} \right] \tag{4.30}$$

This form, while explicit, provides no particularly useful insights; the bath tub end shapes already noted in (4.22) and (4.26) can be deduced directly from (4.30).

Some graphical plots of h_z are presented later. They illustrate the behavior of the present hazard representation in a more understandable fashion than does the formula itself.

D. An Explicit Formula for the Failure Time Distribution

Because x and z are monotonically related through (4.28) we have

$$P\{Z \leq z\} = P\left\{X \leq x = \frac{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{2\alpha(1-\beta z)}\right\}$$

$$= 1 - \exp\left\{-\lambda\left[\frac{(\alpha+\beta)z + \sqrt{(\alpha+\beta)^2 z^2 + 4\alpha z(1-\beta z)}}{2\alpha(1-\beta z)}\right]\right\} \quad (4.31)$$

Again the explicit formula seems unproductive of insights.

V. MATHEMATICAL PROPERTIES OF THE DECREASING FAILURE RATE MODEL

A. The Hazard Behavior.

The expression (4.14) can be applied to deduce the hazard function of the representation (3.8), advertised to produce a decreasing failure rate. There we specified

$$\phi(x) \equiv T(x) = 1 + cx \ , \quad (5.1)$$

and thus, from (4.14),

$$h_w(w(p)) = \frac{\lambda}{(1 + cx(p)) + x(p)c} = \frac{\lambda}{1 + 2cx(p)} \quad (5.2)$$

Qualitative properties follow easily.

(a) If $p \to 0$, $x(p) \downarrow 0$, and

$$h_w(w(p)) \sim \lambda \quad (5.3)$$

Thus the hazard is approximately λ for small z.

(b) if $p \to 1$, $x(p) \uparrow \infty$, and

$$w(p) \sim c[x(p)]^2$$

so

$$h_w(w(p)) \sim \frac{\lambda}{2\sqrt{cw(p)}} , \tag{5.5}$$

which clearly decreases, as claimed. It may be inferred that the distribution of W appears nearly exponential, but possesses an extraordinarily long right tail--possibly the result of outliers.

B. Explicit Formulas for the Hazard and the Distribution Function

Direct solution of the quadratic equation

$$w = x(1 + cx) = x + cx^2$$

presents

$$x(p) = \frac{\sqrt{1 + 4cw(p)} - 1}{2c} , \tag{5.6}$$

which, when differentiated, leads to

$$h_w(w) = \frac{1}{\sqrt{1 + 4cw}} h_x(x)$$

$$= \frac{\lambda}{\sqrt{1 + 4cw}} \tag{5.7}$$

The distribution function is

$$P\{W \leq w\} = P\left\{X \leq x = \frac{\sqrt{1 + 4cw} - 1}{2c}\right\} \tag{5.8}$$

$$= 1 - \exp\left\{-\lambda\left[\frac{\sqrt{1 + 4cw} - 1}{2c}\right]\right\}$$

This distribution bears a close family resemblance to the Weibull distribution $1 - F(w) = \exp\{-k\sqrt{w}\}$, especially for large (right tail) values of w.

VI. AN ALTERNATIVE "BATHTUB" HAZARD REPRESENTATION

The simple parametric model (3.6) leading to a bathtub-shaped hazard is by no means the only possibility. We next describe another simple approach. It is that of defining a hazard function having an appropriate shape, and then deducing the corresponding distribution function, and a procedure for sampling from it, rather than proceeding in reverse order, as before.

Let the hazard be of the form

$$h(z) = g(z) + \lambda + k(z) \ , \tag{6.1}$$

where $g(z) > 0$ is a decreasing function of z such that $\lim_{x \to \infty} g(z) = 0$, and $k(x)$ is an increasing function of x, such that (preferably) $k(0) = 0$ and $k(\infty) = \infty$. Such a function can yield a bathtubbed hazard.

<u>Example.</u>

$$h(z) = \frac{A}{z+\alpha} + Bz + \lambda \tag{6.2}$$

A, B, α, λ all positive.

Clearly, (6.2) has a generally "bath tub-like" appearance, since

$$h'(z) = -\frac{A}{(z+\alpha)^2} + B \tag{6.3}$$

for if

$$-\frac{A}{\alpha} + B < 0, \quad \text{then} \quad h'(0) < 0 \tag{6.4}$$

while for

$$z > z_0 = \left(\frac{A}{B}\right)^{1/2} - \alpha \tag{6.5}$$

$h(z) > 0$.

Detailed behavior is adjustable by choice of the parameters. Now the distribution function of time to failure, Z, is obtained from (6.2):

$$\begin{aligned} P\{Z > z\} &= \exp\left\{-\int_0^z h(x)\,dx\right\} \\ &= \exp\left\{-\int_0^z \left(\frac{A}{x+\alpha} + Bx + \lambda\right)dx\right\} \\ &= \exp\left\{-\left[A\,\ell n\left(1+\frac{z}{\alpha}\right) + \frac{B}{2}z^2 + \lambda z\right]\right\} \\ &= \left(\frac{\alpha}{\alpha+z}\right)^A \exp\left(-\frac{B}{2}z^2\right)e^{-\lambda z} \\ &= \bar{F}_1(z)\,\bar{F}_2(z)\,\bar{F}_3(z)\,, \end{aligned} \tag{6.6}$$

where

$$\bar{F}_1(z) = \left(\frac{\alpha}{\alpha + z}\right)^A$$

$$\bar{F}_2(z) = \exp\left(-\frac{B}{2} z^2\right) \qquad (6.7)$$

$$\bar{F}_3(z) = e^{-\lambda z} .$$

All of the above are recognized as being the complements of distribution functions. In effect, the distribution of Z is that of the minimum of three independent random variables:

$$P\{Z > z\} = P\{X_1 > z\} \cdot P\{X_2 > z\} \cdot P\{X_3 > z\} , \qquad (6.8)$$

X_i having the distribution $F_i = 1 - \bar{F}_i$ $(i = 1,2,3)$. This fact leads directly to an easy procedure for simulation of Z by simply obtaining the smallest from among the realization of X_1, X_2, and X_3. The advantage of the previous method, based on (3.6) for instance, is that only one realization--that of an exponential in that specific example--leads to the realization of Z. This is not only computationally attractive, but seems to facilitate the application of such Monte Carlo variance reduction techniques as control and antithetic variables cf. Hammersley and Handscomb (1964).

VII. OBTAINING SPECIFIED HAZARD BEHAVIOR BY SIMPLE SAMPLING

The development of the last section illustrates one manner in which hazard behavior may be conveniently represented and simulated. We now show how such behavior may alternatively be obtained by simple simulation, i.e. from one realization of a basic (possibly exponential) random variable.

Refer to (3.5), in which

$$Z = G(X) \tag{7.1}$$

and, if $G(\cdot)$ is monotonically increasing,

$$z(p) = G(x(p)) , \tag{7.2}$$

$z(p)$ and $x(p)$ being the $p \cdot 100\%$ percentiles of Z and X respectively. Then the counterpart to (4.14) that results from differentiation of (7.2) is the expression

$$h_z(z(p)) = h_x(x(p)) \frac{1}{G'(x(p))} = h_x(x(p)) \frac{1}{(dz/dx)} \tag{7.3}$$

Consequently, if one specifies $h_z(z)$ as a suitable function of the "time" z, and specifies the distribution of the stochastic variable X--and hence its hazard, h_x--there results a differential equation for $z(x) \equiv G(x)$:

$$h_z(z) \frac{dz}{dx} = h_x(x) ; \tag{7.4}$$

integration then provides the desired transformation, G. In other words, we seek $z(x)$ satisfying

$$\int_0^z h_z(u)du = \int_0^x h_x(v)dv, \tag{7.5}$$

which can sometimes be carried out in a useful closed form.

Example 7.1. Refer to the example of Section VI, wherein h_z is given by the expression (6.2) and we assume that X is exponential, so h_x is constant. Then in order to determine $G(x) \equiv z(x)$, solve the equation

$$\int_0^z \left[\frac{A}{v+\alpha} + Bv + \lambda\right] dv = \int_0^x du$$

or

$$A \ln(1 + \frac{z}{\alpha}) + \frac{B}{2} z^2 + \lambda z = x \tag{7.6}$$

Closed-form solution of this expression for z in terms of x is of course impossible. One possible approach is purely numerical: find an approximate solution, $z_0(x)$, e.g. the appropriate solution of the quadratic

$$\frac{B}{2} z^2 + \lambda z = x \tag{7.7}$$

and then correct the result by a few Newton-Raphson iterations. In other words, put

$$z_1(x) = \frac{-\lambda + \sqrt{\lambda^2 + 2B\lambda x}}{B} ; \tag{7.8}$$

now apply Newton to obtain an improved solution

$$z_2(x) = z_1 - \frac{A\,\ell n(1 + z_1/\alpha)}{A/(z_1 + \alpha) + Bz_1 + \lambda} \tag{7.9}$$

which will be feasible if $0 < z_2$. The process can be iterated (the numerator will change after the first iteration). If one wishes to use this model it may actually be desirable to start by solving

$$\frac{B}{2} z^2 + (\lambda + \frac{A}{\alpha})z - x = 0 \tag{7.10}$$

for z_1, in which case the numerator will not be as shown in (7.9); convergence may be more rapid.

Example 7.2. Change the hazard representation of the previous example as follows: let

$$h_z(v) = \frac{A}{(v + \alpha)^2} + Bv + \lambda; \quad (A, \alpha, B, \lambda > 0) \tag{7.11}$$

then

$$\int_0^z h_z(v)\,dv = \frac{Az}{\alpha(z + \alpha)} + \frac{B}{2} z^2 + \lambda z \tag{7.12}$$

Now the solution of the equation

$$\frac{Az}{\alpha(z + \alpha)} + \frac{B}{2} z^2 + \lambda z = x \qquad (7.13)$$

can be carried out in closed (if cumbersome) form, since (7.13) is a cubic. Hence an explicit representation of $z(p)$ in terms of an exponential's $x(p)$ can be provided. Perhaps more importantly, a direct, simple, simulation of a bathtubbed random variable is at hand.

VIII. CONCLUSION

The purpose of this report has been to show that rather complex distributional behavior can be represented in terms of simple standard distributions. We have concentrated here upon alterations of the exponential, but it is obvious that other distributions can be treated similarly.

In a later report we will illustrate the use of this technique in data analysis, and in the simulation of non-homogeneous stochastic point processes.

REFERENCES

D. P. Gaver and Chu, B. (1977). Estimating equipment and system availability by use of the jackknife. EPRI Tech. Report; also Naval Postgraduate School Technical Report NPS55-77-4.

Gaver, D. P., Lavenburg, S., and Price, T. (1976). Exploratory analysis of access path length data for a data base management system. IBM J. of Research and Development, Vol. 20 pp. 449-464.

Hammersley, J., and Handscomb, D.C. (1964). Monte Carlo Methods. Wiley, New York.

Parzen, M. (1978). Nonparametric statistical data modeling. Paper presented at National Meeting American Statistical Association, San Diego, August 1978.

Tukey, J. W. (1976). Usable resistant/robust technique of analysis (p. 11). Proc. of the First ERDA Statistical Symposium. Battelle Pacific Northwest Laboratories, Richland, Washington.

DISTRIBUTION LIST

	No. of Copies
STATISTICS AND PRCBABILITY FROGRAM CFFICE OF NAVAL RESEARCH CODE 436 ARLINGTON VA 22217	1
CFFICE OF NAVAL RESEARCH NEW YORK AREA CFFICE 715 BROADWAY - 5TH FLOOR ATTN: DR. ROBER GRAFTON NEW YORK, NY 10003	1
DIRECTOR CFFICE OF NAVAL RESEARCH BRANCH OFF 536 SOUTH CLARK STREET ATTN: DEPUTY AND CHIEF SCIENTIST CHICAGO, IL 60605	1
LIBRARY NAVAL OCEAN SYSTEMS CENTER SAN DIEGO CA 92152	1
NAVY LIBRARY NATIONAL SPACE TECHNOLOGY LAB ATTN: NAVY LIBRARIAN BAY ST. LOUIS MS 39522	1
NAVAL ELECTRONIC SYSTEMS COMMAND NAVELEX 320 NATIONAL CENTER NO. 1 ARLINGTON VA 20360	1
DIRECTOR NAVAL REAEARCH LABORATORY ATTN: LIBRARY (ONRL) CODE 2029 WASHINGTON, D.C. 20375	1
DEFENSE DOCUMENTATION CENTER CAMERON STATION ALEXANDRIA VIRGINIA 22314	2
TECHNICAL INFORMATION DIVISION NAVAL RESEARCH LABORATORY WASHINGTON, D. C. 20375	1

No. of Copies

OFFICE OF NAVAL RESEARCH SAN FRANCISCO AREA OFFICE 760 MARKET STREET SAN FRANCISCO CALIFORNIA 94102	1
TECHNICAL LIBRARY NAVAL ORDNANCE STATION INDIAN HEAD MARYLAND 20640	1
NAVAL SHIP ENGINEERING CENTER PHILADELPHIA DIVISION TECHNICAL LIBRARY PHILADELPHIA PENNSYLVANIA 19112	1
BUREAU OF NAVAL PRESONNEL DEPARTMENT OF THE NAVY TECHNICAL LIBRARY WASHINGTON D. C. 20370	1
LIBRARY CODE 0212 NAVAL POSTGRADUATE SCHOOL MONTEREY CALIFORNIA 93940	2
PROF. M. ABDEL-HAMEED DEPARTMENT OF MATHEMATICS UNIVERSITY OF NORTH CAROLINA CHARLOTTE NC 28223	1
PROF. T. W. ANDERSON DEPARTMENT OF STATISTICS STANFORD UNIVERSITY STANFORD , CALIFORNIA 94305	1
PROF. F. J. ANSCOMBE DEPARTMENT OF STATISTICS YALE UNIVERSITY NEW HAVEN CONNECTICUT 06520	1
PROF. L. A. ARDIAN INSTITUTE OF INDUSTRIAL ADMINISTRATION UNION COLLEGE SCHENECTADY , NEW YORK 12308	1

No. of Copies

PROF. C. R. BAKER
DEPARTMENT OF STATISTICS
UNIVERSITY OF NOTRH CAROLINA
CHAPEL HILL,
NORTH CAROLINA
27514
1

PROF. R. E. BECHHOFER
DEPARTMENT OF OPERATIONS RESEARCH
CORNELL UNIVERSITY
ITHACA
NEW YORK 14850
1

PROF. N. J. BERSHAD
SCHOOL OF ENGINEERING
UNIVERSITY OF CALIFORNIA
IRVINE ,
CALIFORNIA
92664
1

P. J. BICKEL
DEPARTMENT OF STATISTICS
UNIVERSITY OF CALIFORNIA

BERKELEY , CALIFORNIA
94720
1

PROF. H. W. BLOCK
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF PITTSBURGH
PITTSBURGH
PA
15260
1

PROF. JOSEPH BLUM
DEPT. OF MATHEMATICS, STATISTICS
AND COMPUTER SCIENCE
THE AMERICAN UNIVERSITY
WASHINGTON
DC
20016
1

PROF. R. A. BRADLEY
DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY

TALLAHASSEE , FLORIDA 32306
1

PROF. R. E. BARLOW
OPERATIONS RESEARCH CENTER
COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKLEY
CALIFORNIA 94720
1

MR. C. N BENNETT
NAVAL COASTAL SYSTEMS LABORATORY
CODE P761
PANAMA CITY,
FLORIDA
32401
1

No. of Copies

PROF. U. N. BHAT — 1
COMPUTER SCIENCE / OPERATIONS
RESEARCH CENTER
SOUTHERN METHODIST UNIVERSITY
DALLAS ,
TEXAS 75275

PROF. W. R. BLISCHKE — 1
DEPT. OF QUANTITATIVE
BUSINESS ANALYSIS
UNIVERSITY OF SOUTHERN CALIFORNIA
LOS ANGELES, CALIFORNIA
90007

DR. DERRILL J. BORDELON — 1
NAVAL UNDERWATER SYSTEMS CENTER
CODE 21
NEWPORT
RI
02840

J. E. BOYER JR — 1
DEPT. OF STATISTICS
SOUTHERN METHODIST UNIVERSITY
DALLAS
TX
75275

DR. J. CHANDRA — 1
U. S. ARMY RESEARCH
P. O. BOX 12211
RESEARCH TRIANGLE PARK ,
NORTH CAROLINA
27706

PROF. H. CHERNOFF — 1
DEPT. OF MATHEMATICS
MASS INSTITUTE OF TECHNOLOGY
CAMBRIDGE,
MASSACHUSETTS 02139

PROF. C. DERMAN — 1
DEPARTMENT OF CIVIL ENGINEERING
AND ENGINEERING MECHANICS
COLUMBIA UNIVERSITY
NEW YORK
NEW YORK
10027

PROF. R. L. DISNEY — 1
VIRGINIA POLYTECHNIC INSTITUTE
AND STATE UNIVERSITY
DEPT. OF INDUSTRIAL ENGINEERING
AND OPERATIONS RESEARCH
BLACKSBURG, VA
24061

MR. J. DOWLING — 1
DEFENSE LOGISTICS STUDIES
INFORMATION EXCHANGE
ARMY LOGISTICS MANAGEMENT CENTER
FORT LEE ,
VIRGINIA 20390

No. of Copies

PROF J. D. ESARY
DEPT. OF OPERATIONS RESEARCH AND
ADMINISTRATIVE SCIENCES
NAVAL POSTGRADUATE SCHOOL
MONTEREY ,
CALIFORNIA 93940

1

DR. M. J. FISCHER
DEFENSE COMMUNICATIONS AGENCY
1860 WIEHLE AVENUE
RESTON
VIRGINIA
22070

1

MR. GENE H. GLEISSNER
APPLIED MATHEMATICS LABORATORY
DAVID TAYLOR NAVAL SHIP RESEARCH
AND DEVELOPMENT CENTER
BETHESDA
MD 20084

1

PROF. S. S. GUPTA
DEPARTMENT OF STATISTICS
PURDUE UNIVERSITY
LAFAYETTE
INDIANA 47907

1

PROF. D. L. HANSON
DEPT OF MATH. SCIENCES
STATE UNIVERSITY OF NEW YORK,
BINGHAMTON
BINGHAMTON
NY 13901

1

PROF. F. J. HARRIS
DEPT. OF ELECTRICAL ENGINEERING
SAN DIEGO STATE UNIVERSITY
SAN DIEGO
CA
92182

1

PROF. L. H. HERBACH
DEPT. OF OPERATIONS RESEARCH AND
SYSTEMS ANALYSIS
POLYTECHNIC INSTITUTE OF NEW YORK
BROOKLYN
NY 11201

1

PROF. M. J. HINICH
DEPARTMENT OF ECONOMICS
VIRGINIA POLYTECHNIC INSTITUTE
AND STATE UNIVERSITY
BLACKSBURG ,
VIRGINIA 24061

1

No. of Copies

PROF. W. M. HIRSCH 1
INSTITUTE OF MATHEMATICAL SCIENCES
NEW YORK UNIVERSITY
NEW YORK
NEW YORK 10453

PROF. J. B. KADANE 1
DEPARTMENT OF STATISTICS
CARNEGIE-MELLON
PITTSBURGH ,
PENNSYLVANIA
15213

DR. RICHARD LAU 1
DIRECTOR
OFFICE OF NAVAL RESEARCH BRANCH OFF
1030 EAST GREEN STREET
PASADENA
CA 91101

DR. A. R. LAUFER 1
DIRECTOR
OFFICE OF NAVAL RESEARCH BRANCH OFF
1030 EAST GREEN STREET
PASADENA
CA 91101

PROF. M. LEADBETTER 1
DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL
NORTH CAROLINA 27514

DR. J. S. LEE 1
J. S. LEE ASSOCIATES, INC.
2001 JEFFERSON DAVIS HIGHWAY
SUITE 802
ARLINGTON
VA 22202

PROF. L. C. LEE 1
DEPARTMENT OF STATISTICS
VIRGINIA POLYTECHNIC INSTITUTE
AND STATE UNIVERSITY
BLACKSBURG
VA 24061

PROF. R. S. LEVENWORTH 1
DEPT. OF INDUSTRIAL AND SYSTEMS
ENGINEERING
UNIVERSITY OF FLORIDA
GAINSVILLE ,
FLORIDA 32611

No. of Copies

FRCF. P. A. W. LEWIS 1
CEPT. CF CFERATICNS RESEARCH AND
ADMINISTRATIVE SCIENCES
NAVAL FCST GRACLATE SCFCOL
MONTEREY , CALIFORNIA
93940

FRCF G. LIEBERMAN 1
STANFORC INIVERSITY
CEFARTMENT CF OPERATICNS RESEARCH

STANFORD CALIFORNIA 94305

DR. JAMES R. MAAR 1
NATICNAL SECURITY AGENCY
FORT MEADE , MARYLAND
20755

FRCF. R. W. MACSEN 1
DEPARTMENT CF STATISTICS
UNIVERSITY OF MISSCURI
COLUMBIA
MO
65201

DR. N. R. MANN 1
SCIENCE CENTER
ROCKWELL INTERNATICNAL CORFCRATION
P.C. BOX 1085
THCUSAND CAKS ,
CALIFORNIA 9136C

CR. W. H. MARLCW 1
PROGRAM IN LOGISTICS
THE GEORGE WASHINGTCN UNIVERSITY
707 22NC STREET , N. W.
WASHINGTCN , D. C.
20037

PROF. E. MASRY 1
DEPT. APPLIEC PHYSICS AND
INFORMATICN SERVICE
UNIVERSITY OF CALIFORNIA
LA JCLLA
CALIFORNIA
92093

CR. BRCCE J. MCCONALD 1
SCIENTIFIC DIRECTCR
SCIENTIFIC LIAISON GROUP
OFFICE CF NAVAL RESEARCH
AMERICAN EMBASSY - TOKYC
AFC SAN FRANCISCO
96503

PROF. J. A. MUCKSTADT 1
DEPT. CF CPERATICNS RESEARCH
CCRNELL UNIVERSITY
ITHACA,
NEW YCRK
19850

No. of Copies

DR. JANET M. MYHRE 1
THE INSTITUTE OF DECISION SCIENCE
FOR BUSINESS AND PUBLIC POLICY
CLAREMONT MEN'S COLLEGE
CLAREMONT
CA 91711

MR. H. NISSELSON 1
BUREAU OF THE CENSUS
ROOM 2025
FEDERAL BUILDING 3
WASHINGTON ,
D. C. 2033

MISS B. S. ORLEANS 1
NAVAL SEA SYSTEMS COMMAND
(SEA 03F)
RM 10S08
ARLINGTON VIRGINIA 20360

PROF. D. B OWEN 1
DEPARTMENT OF STATISTICS
SOUTHERN METHODIST UNIVERSITY
DALLAS
TEXAS
75222

PROF. E. PARZEN 1
STATISTICAL SCIENCE DIVISION
STATE UNIVERSITY OF NEW YORK
AT BUFFALO
AMHERST
NEW YORK 14226

DR. A. PETRASOVITS 1
ROOM 207B , FOOD AND DRUG BLDG.
TUNNEY'S PASTURE
OTTOWA , ONTARIO K1A-OL2 ,
CANADA

PROF. S. L. PHOENIX 1
SIBLEY SCHOOL OF MECHANICAL AND
AEROSPACE ENGINEERING
CORNELL UNIVERSITY
ITHACA
NY 14850

DR. A. L. POWELL 1
DIRECTOR
OFFICE OF NAVAL RESEARCH BRANCH OFF
495 SUMMER STREET
BOSTON
MA 02210

MR. F. R. PRIORI 1
CODE 224 OPERATIONSL TEST AND ONRS
EVALUATION FORCE (OPTEVFOR)
NORFOLK ,
VIRGINIA
20360

No. of Copies

PROF. M. L. PURI
DEPT. OF MATHEMATICS
P.O. BOX F
INDIANA UNIVERSITY FOUNDATION
BLOOMINGTON
IN 47401

1

PROF. H ROBBINS
DEPARTMENT OF MATHEMATICS
COLUMBIA UNIVERSITY
NEW YORK,
NEW YORK 10027

1

PROF. M ROSENBLATT
DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA SAN DIEGO
LA JOLLA
CALIFORNIA
92093

1

PROF. S. M. ROSS
COLLEGE OF ENGINEERING
UNIVERSITY OF CALIFORNIA
BERKELEY
CA
94720

1

PROF. I RUBIN
SCHOOL OF ENGINEERING AND APPLIED
SCIENCE
UNIVERSITY OF CALIFORNIA
LOS ANGELES ,
CALIFORNIA 90024

1

PROF. I. R. SAVAGE
DEPARTMENT OF STATISTICS
YALE UNIVERSITY
NEW HAVEN,
CONNECTICUT
06520

1

PROF. L. L. , SCHARF JR
DEPARTMENT OF ELECRICAL ENGINEERING
COLORADO STATE UNIVERSITY
FT. COLLINS ,
COLORADO
80521

1

PROF. R. SERFLING
DEPARTMENT OF STATISTICS
FLORIDA STATE UNIVERSITY

TALLAHASSEE FLORIDA 32306

1

PROF. W. R. SCHUCANY
DEPARTMENT OF STATISTICS
SOUTHERN METHODIST UNIVERSITY
DALLAS ,
TEXAS
75222

1

No. of Copies

PROF. D. C. SIEGMUND
DEPT. OF STATISTICS
STANFORD UNIVERSITY
STANFORD
CA
94305 1

PROF. M. L. SHOOMAN
DEPT. OF ELECTRICAL ENGINEERING
POLYTECHNIC INSTITUTE OF NEW YORK
BROOKLYN,
NEW YORK
11201 1

PROF. N. SINGPURWALLA
DEPT. OF OPERATIONS RESEARCH
THE GEORGE WASHINGTON UNIVERSITY
707 22ND ST . N. W.
WASHINGTON, D. C.
20052 1

DR. A. L. SLAFKOSKY
SCIENTIFIC ADVISOR
COMMANDANT OF THE MARINE CORPS
WASHINGTON ,
D. C.
20380 1

MR. CHARLES S. SMITH
OASD (I&L),
PENTAGON
WASHINGTON
DC
20301 1

DR. D. E. SMITH
DESMATICS INC.
P.O. BOX 618
STATE COLLEGE
PENNSYLVANIA
16801 1

PROF. W. L. SMITH
DEPARTMENT OF STATISTICS
UNIVERSITY OF NORTH CAROLINA
CHAPEL HILL
NORTH CAROLINA 27514 1

PROF. H SOLOMON
DEPARTMENT OF STATISTICS
STANFORD UNIVERSITY
STANFORD ,
CALIFORNIA
94305 1

MR. GLENN F. STAHLY
NATIONAL SECURITY AGENCY
FORT MEADE ,
MARYLAND 20755 1

No. of Copies

MR. DAVID A. SWICK 1
ADVANCED PROJECTS GROUP
CODE 8103
NAVAL RESEARCH LAB.
WASHINGTON
DC 20375

MR. WENDELL G. SYKES 1
ARTHUR D. LITTLE, INC.
ACORN PARK
CAMBRIDGE
MA 02140

PROF. J. R. THOMPSON 1
DEPARTMENT OF MATHEMATICAL SCIENCE
RICE UNIVERSITY
HOUSTON,
TEXAS
77001

PROF. W. A. THOMPSON 1
DEPARTMENT OF STATISTICS
UNIVERSITY OF MISSOURI
COLUMBIA ,
MISSOURI
65201

PROF. F. A. TILLMAN 1
DEPT. OF INDUSTRIAL ENGINEERING
KANSAS STATE UNIVERSITY
MANHATTAN
KS 66506

PROF. A . F . VEINOTT 1
DEPARTMENT OF OPERATIONS RESEARCH
STANFORD UNIVERSTITY
STANFORD
CALIFORNIA
94305

DANIEL H. WAGNER 1
STATION SQUARE ONE
PAOLI , PENNSYLVANIA
19301

PROF. GRACE WAHBA 1
DEPT. OF STATISTICS
UNIVERSITY OF WISCONSIN
MADISON
WI 53706

	No. of Copies
PROF. K. T. WALLENIUS DEPARTMENT OF MATHEMATICAL SCIENCES CLEMSON UNIVERSITY CLEMSON, SOUTH CAROLINA 29631	1
PROF. BERNARD WIDROW STANFORD ELECTRONICS LAB STANFORD UNIVERSITY STANFORD CA 94305	1
PROF. G. E. WHITEHOUSE DEPT. OF INDUSTRIAL ENGINEERING LEHIGH UNIVERSITY BETHLEHEM PA 18015	1
PROF. S. ZACKS DEPT. OF MATHEMATICS AND STATISTICS CASE WESTERN RESERVE UNIVERSITY CLEVELAND., OHIO 44106	1
PROF. M. ZIA-HASSAN DEPARTMENT OF INDUSTRIAL AND SYSTEMS ENGINEERING ILLINOIS INSTITUTE OF TECHNOLOGY CHICAGO IL 60616	1
HEAD, MATH. SCI SECTION NAT. SCIENCE FOUNDATION 1800 G STREET, N.W. WASHINGTON, D.C. 20550	1
PROF. A. F. ANDRUS DEPT. OF OR NAVAL POSTGRADUATE SCHOOL MONTEREY CALIFORNIA 93940	1
PROF. D. R. BARR DEPT. OF OR NAVAL POSTGRADUATE SCHOOL MONTEREY CALIFORNIA 93940	1

No. of Copies

PROF. PETER BLOOMFIELD
STATISTICAL DEPT.
PRINCETON UNIVERSITY
PRINCETON, N. J. 08540 — 1

PROF. G. G. BROWN
DEPT. OF OR
NAVAL POSTGRADUATE SCHOOL
MONTEREY
CALIFORNIA 93940 — 1

R. W. BUTTERWORTH
SYSTEMS EXPLORATION
WEBSTER ST.
MONTEREY
CALIFORNIA 93940 — 1

DR. JAMES CAPRA
7218 DELFIELD STREET
CHEVY CHASE
MARYLAND 20015 — 1

DR. D.R. COX
DEPT. OF MATHEMATICS
IMPERIAL COLLEGE
LONDON SW7
ENGLAN — 1

ECON. AND MAN. SCI. CTR.
NORTHWESTERN UNIV.
EVANSTON
ILLINOIS 602 01 — 1

MAN. SCE. RES. CTR.
FACULTY OF COM. AND BUS. ADMIN.
UNIV. OF BRITISH COLUMBIA
VANCOUVER
BRITISH COLUMBIA V6T 1W5 CANADA — 1

DR. M. DWASS
MATH. DEPT.
NORTHWESTERN UNIV.
EVANSTON
ILLINOIS 60201 — 1

No. of Copies

DR. R. ELASHOFF
BIOMATHEMATICS
UNIV. OF CALIF.
LOS ANGELES
CALIFORNIA
90024 — 1

PROF. GEORGE S. FISHMAN
UNIV. OF NORTH CAROLINA
CUR. IN OR AND SYS. ANALYSIS
PHILLIPS ANNEX
CHAPEL HILL, NORTH CAROLINA
20742 — 1

DR. R. GNANADESIKAN
BELL TELEPHONE LAB
HOLMDEL, N. J.
07733 — 1

DR. A. J. GOLDMAN
CHIEF, OR
DIV. 205.02, ADMIN. A428
U.S. DEPT. OF COMMERCE
WASHINGTON, D.C.
20234 — 1

DR. H. HIGGINS
53 BONN 1, POSTFACH 589
NASSESTRASSE 2
WEST GERMANY — 1

DR. P. T. HOLMES
DEPT. OF MATH.
CLEMSON UNIV.
CLEMSON
SOUTH CAROLINA
29631 — 1

DR. J. A. HOOKE
BELL TELEPHONE LABS
HOLMDEL
NEW JERSEY
07733 — 1

DR. ROBERT HOOKE
MATH. DEPT.
WESTINGHOUSE RES. LABS
CHURCHILL BORO
PITTSBURGH, PENNSYLVANIA
15235 — 1

DR. D. L. IGLEHART
DEPT. OF O.R.
STANFORD UNIV.
STANFORD
CALIFORNIA
94305 — 1

No. of Copies

DR. PATRICIA JACOBS 1
OR DEPT.
NAVAL POSTGRADUATE SCHOOL
MONTEREY
CALIFORNIA
93940

DR. H. KOBAYASHI 1
IBM
YORKTOWN HEIGHTS
NEW YORK
10598

DR. JOHN LEHOCZKY 1
STATISTICS DEPARTMENT
CARNEGIE-MELLON UNIVERSITY
PITTSBURGH
PENNSYLVANIA
15213

LIBRARY 1
CODE 55
NAVAL POSTGRADUATE SCHOOL
MONTEREY
CALIFORNIA 93940

DR. A. LEMOINE 1
1020 GUINDA ST.
PALO ALTO,
CALIFORNIA
94301

DR. J. MACQUEEN 1
UNIV. OF CALIF.
LOS ANGELES
CALIFORNIA
90024

PROF. K. T. MARSHALL 1
DEPT. OF OR
NAVAL POSTGRADUATE SCHOOL
MONTEREY
CALIFORNIA
93940

DR. M. MAZUMDAR 1
MATH. DEPT.
WESTINGHOUSE RES. LABS
CHURCHILL BORO
PITTSBURGH
PENNSYLVANIA 15235

DR. LEON F. MCGINNIS 1
SCHOOL OF IND. AND SYS. ENG.
GEORGIA INST. OF TECH.
ATLANTA
GEORGIA
30332

No. of Copies

DR. D. R. MCNEIL
DEPT. OF STATISTICS
PRINCETON UNIV.
PRINCETON
NEW JERSEY
08540 — 1

PROF. P. R. MILCH
DEPT. OF OR
NAVAL POSTGRADUATE SCHOOL
MONTEREY
CALIFORNIA
93940 — 1

DR. F. MOSTELLER
STAT. DEPT.
HARVARD UNIV.
CAMBRIDGE
MASSACHUSETTS
02139 — 1

PROF. R. R. READ
DEPT. OF OR
NAVAL POSTGRADUATE SCHOOL
MONTEREY
CALIFORNIA
93940 — 1

DR. M. REISER
IBM
THOMAS J. WATSON RES. CTR.
YORKTOWN HEIGHTS
NEW YORK
10598 — 1

PROF. F. R. RICHARDS
DEPT. OF OR
NAVAL POSTGRADUATE SCHOOL
MONTERY
CALIFORNIA
93940 — 1

DR. J. RIORDAN
DEPT. OF MATHEMATICS
ROCKEFELLER UNIV.
NEW YORK
NEW YORK
10021 — 1

DR. LINUS SCHRAGE
UNIV. OF CHICAGO
GRAD. SCHOOL OF BUS.
5836 GREENWOOD AVE.
CHICAGO, ILLINOIS
60637 — 1

	No. of Copies
DR. PAUL SCHWEITZER THOMAS J. WATSON RESEARCH CTR. POST OFFICE BOX 218 YORKTOWN HEIGHTS NEW YORK 10598	1
DR. RICHARD SORENSON CODE 303 NPRDC 271 CATALINA BLVD. SAN DIEGO CALIFORNIA 92152	1
PROF. M. G. SOVEREIGN DEPT. OF OR NAVAL POSTGRADUATE SCHOOL MONTEREY CALIFORNIA 93940	1
DR. V. SRINIVASAN GRADUATE SCHOOL OF BUSINESS STANFORD UNIVERSITY STANFORD CALIFORNIA 94305	1
DR. R. M. STARK STATISTICS AND COMPUTER SCI. UNIV. OF DELAWARE NEWARK DELAWARE 19711	1
PROF. RICHARD VANSLYKE RES. ANALYSIS CORP. BEECHWOOD OLD TAPPEN ROAD GLEN COVE, NEW YORK 11542	1
PROF. JOHN W. TUKEY FINE HALL PRINCETON UNIV. PRINCETON NEW JERSEY 08540	1
DR. THOMAS C. VARLEY OFFICE OF NAVAL RESEARCH CODE 434 ARLINGTON VA 22217	1
PROF. G. WATSON FINE HALL PRINCETON UNIV. PRINCETON NEW JERSEY 08540	1

	No. of Copies
Dr. Roy Welsch M.I.T., Sloan School Cambridge, MA 02139	1
Dean of Research 012 Naval Postgraduate School Monterey, Ca. 93940	1
Professor D. P. Gaver Code 55Gv Naval Postgraduate School Monterey, Ca. 93940	50
Professor Peter Lewis Code 55Lw Naval Postgraduate School Monterey, Ca. 93940	1
R. J. Stampfel Code 55 Naval Postgraduate School Monterey, Ca. 93940	1
Lt. Mustafa Acar, Turkish Navy Kasimpasa, Kadimehmet mah. Akide Sok. No. 7/1, Istanbul Turkey	5
Deniz Kuvvetleri Komutanliği Bakanliklar, Ankara, Turkey	1
Orta-Dogu Teknik Universitesi Yon-Eylem Arastirmasi Bolumu Ankara, Turkey	1